MA1 – 8.1.2021 (a) (vzor)

1. $$\int \left(\frac{\sin x \cos x}{1+\sin^2 x} + \frac{7\sqrt{x}-20}{2x(x-6\sqrt{x}+10)} \right) dx = I_1 + I_2$$

a) integrál existuje v $(0,+\infty)$ – def. spojitost integrandu v I 0,5b

b) $$I_1 = \int \frac{\sin x \cos x}{1+\sin^2 x}\, dx \underset{1VS}{=} \left| \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right| = \int \frac{t}{1+t^2}\, dt =$$

2b

$$= \frac{1}{2} \int \frac{2t}{1+t^2}\, dt \underset{1VS}{=} \frac{1}{2} \ln(1+t^2) + C = \frac{1}{2} \ln(1+\sin^2 x) + C$$

1,5b

(nebo, hned $\underset{1VS}{=}$ $$\left| \begin{array}{l} 1+\sin^2 x = t \\ 2\sin x \cos x\, dx = dt \end{array} \right| = \frac{1}{2} \int_{t>0} \frac{1}{t}\, dt = \frac{1}{2} \ln(1+\sin^2 x) + C$$

c) $$I_2 = \int \frac{7\sqrt{x}-20}{2x(x-6\sqrt{x}+10)}\, dx \underset{2VS}{=} \left| \begin{array}{l} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\, dt \end{array} \right| = \int \frac{7t-20}{2t^2(t^2-6t+10)} \cdot 2t\, dt =$$

2b subst.

$$= \int \frac{7t-20}{t(t^2-6t+10)}\, dt = -2 \int \frac{1}{t}\, dt + \int \frac{2t-5}{t^2-6t+10}\, dt =$$

$$= -2\ln|t| + \int \frac{2t-6}{t^2-6t+10}\, dt + \int \frac{1}{(t-3)^2+1}\, dt =$$

$$-2\ln t + \ln(t^2-6t+10) + \operatorname{arctg}(t-3) + C =$$

$$= -2\ln\sqrt{x} + \ln(x-6\sqrt{x}+10) + \operatorname{arctg}(\sqrt{x}-3) + C, \quad C \in \mathbb{R},\ x \in (0,+\infty)$$

0,5b + 1,5b integrace

Rozklad: $$\frac{7t-20}{t(t^2-6t+10)} = \frac{A}{t} + \frac{Bt+C}{t^2-6t+10}$$

$$7t-20 = A(t^2-6t+10) + Bt^2 + Ct, \text{ tj.}$$

u t^2: $A + B = 0 \Rightarrow B = 2$

u t: $-6A + C = 7 \Rightarrow C = 7+6A = 7-12 = -5$

u t^0: $10t = -20 \Rightarrow A = -2$

2b rozklad

② $f(x) = (x+2)e^{\frac{1}{x}}$

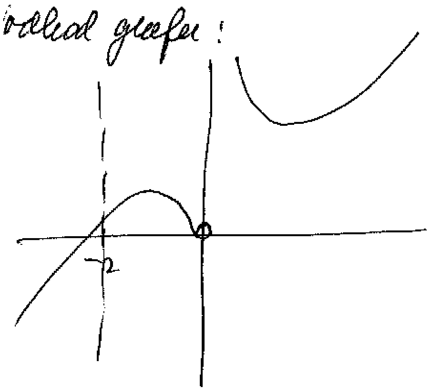
a) f je def. v $(-\infty,0)\cup(0,+\infty)$, zde je spojitá, $f(x)=0 \Leftrightarrow x=-2$, 1

$f(x)>0$ v $(-2,0)\cup(0,+\infty)$, $f(x)<0$ v $(-\infty,-2)$,

$\lim\limits_{x\to\pm\infty} (x+2)e^{\frac{1}{x}} = „\pm\infty \cdot e^{0}" = \pm\infty$ } 1 náčrt grafu:

$\lim\limits_{x\to 0+} (x+2)e^{\frac{1}{x}} = „2\cdot e^{+\infty}" = +\infty$ } 1

$\lim\limits_{x\to 0-} (x+2)e^{\frac{1}{x}} = „2\cdot e^{-\infty}" = 0$



b) $f'(x) = e^{\frac{1}{x}} + (x+2)\,e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) =$

$= \frac{1}{x^2}\, e^{\frac{1}{x}}\left(x^2-x-2\right)$; 1b f'

$f'(x)=0 \Leftrightarrow (x-2)(x+1)=0 \Leftrightarrow$
$x=-1,\ x=2$

f': + (↗) -1 − (↘) 0 − (↘) 2 + (↗)

1b (↗ ↘)

tedy v bodě $x=-1$ je ostré lok. maximum, $f(-1) = e^{-1}$
— " — $x=2$ je ostré lok. minimum (0,5)' $f(2) = 4\sqrt{e}$

globální extrémy fce nemá (limity $\pm\infty$) (0,5) 1b-extrémy

c) $f''(x) = \left(e^{\frac{1}{x}}\left(1-\frac{1}{x}-\frac{2}{x^2}\right)\right)' = e^{\frac{1}{x}}\left(1-\frac{1}{x}-\frac{2}{x^2}\right)\cdot\left(-\frac{1}{x^2}\right) + e^{\frac{1}{x}}\left(\frac{1}{x^2}+\frac{4}{x^3}\right) =$

$= e^{\frac{1}{x}}\cdot\frac{1}{x^4}\left(2+x-\cancel{x^2}+\cancel{x^2}+4x\right) = e^{\frac{1}{x}}\cdot\frac{1}{x^4}(5x+2)$ 1,5 $f''(x)$

$f''(x)=0 \Rightarrow x=-\frac{2}{5}$

− (∩) $-\frac{2}{5}$ + (∪) 0 + (∪)   2

infl. ∩ ∪ 1,5b

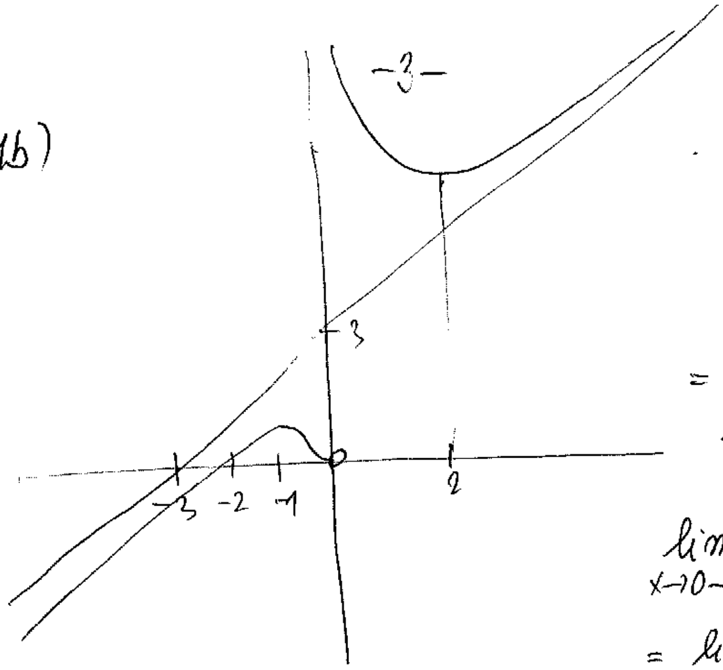
$\Rightarrow$ v $x=-\frac{5}{2}$ je inflexe

d) asymptoty grafu: (i) $x=0$ svislá $\left(\lim\limits_{x\to 0+} f(x) = +\infty\right)$

(ii) šikmé? $\lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{(x+2)e^{\frac{1}{x}}}{x} = \lim\limits_{x\to\pm\infty}\left(1+\frac{2}{x}\right)\cdot e^{\frac{1}{x}} = 1 = a$ 1b

$b = \lim\left(f(x)-x\right) = \lim\limits_{x\to\pm\infty}\left((x+2)e^{\frac{1}{x}}-x\right) = \lim\limits_{x\to\pm\infty}\left(2e^{\frac{1}{x}} + x\left(e^{\frac{1}{x}}-1\right)\right) =$ „$\pm\infty\cdot 0$"

$= \lim\limits_{x\to\pm\infty}\left(2e^{\frac{1}{x}} + \frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}\right) = 2+1 = 3$ („l'Hospital") graf: (1b)



pozn.

$$\lim_{x\to 0-} f'(x) = \lim_{x\to 0-} \frac{1}{x^2} e^{\frac{1}{x}} (x^2 - x - 2) = 0,$$

$$\lim_{x\to 0-} \frac{1}{x^2} e^{\frac{1}{x}} = \text{"}\infty \cdot 0\text{"} = \lim_{t\to +\infty} e^{-t} \cdot t^2 = \text{"}0 \cdot \infty\text{"} = \lim_{t\to +\infty} \frac{t^2}{e^t} = 0$$

③ Obsah omezené oblasti v rovině, ohraničené grafy fcí $y = \ln x$, $y = \ln^2 x$

1) $\ln x = \ln^2 x \iff \ln x = 0 \vee \ln x = 1 \iff x = 1 \vee x = e$;

2) $0 < \ln x < 1$ v $(1, e) \Rightarrow \ln^2 x < \ln x$ pro $x \in (1, e)$.

$$S(\omega) = \int_1^e (\ln x - \ln^2 x)\, dx = 1 - (e - 2) = \underline{\underline{3 - e}}$$

"model" 3b (1,5b – meze, 1,5b $\ln x > \ln^2 x$)

1,5b $\int_1^e \ln x\, dx = [x \ln x - x]_1^e = (e - e) - (-1) = 1$

2,5b $\int_1^e \ln^2 x\, dx = \left| \begin{array}{l} u' = 1 \quad u = x \\ v = \ln^2 x, v' = 2 \ln x \cdot \frac{1}{x} \end{array} \right| = [x \ln^2 x]_1^e - 2 \int_1^e x \ln x \cdot \frac{1}{x} dx =$

$= [x \ln^2 x]_1^e - 2 \cdot 1$

$= e - 2$

integrace 4b

(4) $y' = \frac{2x}{1-x^2}(y-1)$: $\underbrace{x\in(-\infty,-1)}_{I_1}$, $\underbrace{x\in(-1,1)}_{I_2}$, $\underbrace{x\in(1,+\infty)}_{I_3}$

(i) $y(x)=1$, $x\in I_1, x\in I_2, x\in I_3$ – stac. řešení   1b stac. řešení

(ii) $y\neq 1$: separace   1b separace (−0,5, když chybí $y(x)\neq 1$)

$$\frac{dy}{y-1} = -\frac{2x}{x^2-1}\,dx$$ integrace 1+1

$$\ln|y-1| = -\ln|x^2-1| + C$$

$$|y-1| = \frac{e^C}{|x^2-1|}\ , \text{ tj. } \quad y = 1+\frac{k}{x^2-1}\ ,\ k\neq 0,\ x\neq\pm 1$$

(i) a (ii): $y_{ob}(x) = 1 + \frac{k}{x^2-1}$, $k\in\mathbb{R}$, $x\in(-\infty,-1)$, $x\in(-1,1)$, $x\in(1,+\infty)$

„vzorec" $y(x)$ .. 3b (−1b bez intervalů)

(abs. hodnota – odstranění 1b)

Počáteční úloha:

(i) $y(2)=3$: $x\in(1,+\infty)$

$1+\frac{k}{3} = 3 \Rightarrow k=6$ $y(x) = 1+\frac{6}{x^2-1}$, $x\in(1,+\infty)$ 0,5b

(ii) $y(0)=0$, $x\in(-1,1)$

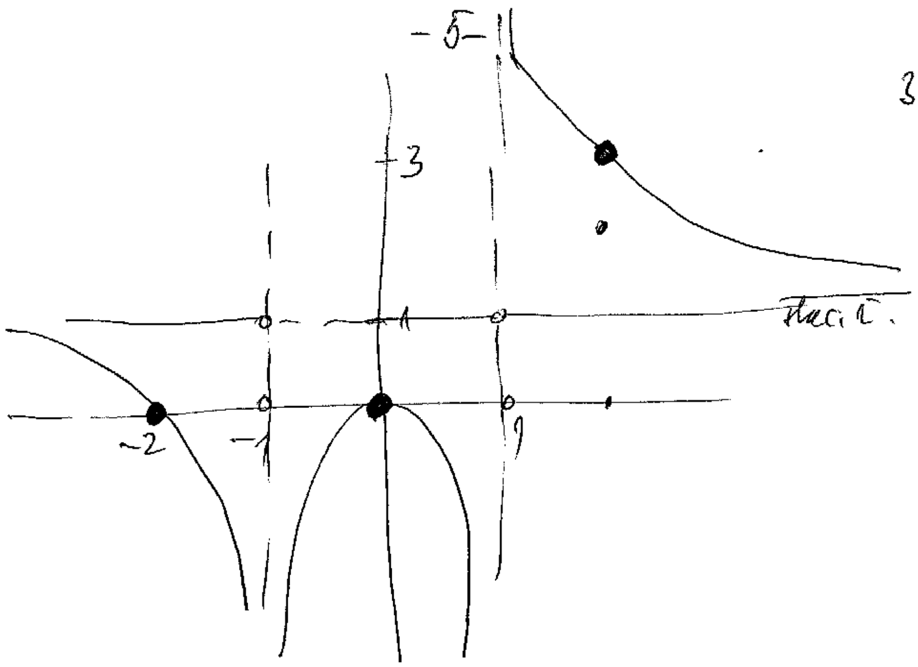
$0 = 1-k \Rightarrow k=1$ $y(x) = 1+\frac{1}{x^2-1}$, $x\in(-1,1)$ 0,5b

(iii) $y(-2)=0$, $x\in(-\infty,-1)$

$0 = 1+\frac{k}{3} \Rightarrow k=-3$ $y(x) = 1-\frac{3}{x^2-1}$, $x\in(-\infty,-1)$ 0,5b

Graf:

graf:



3 x 0,5b

Otázky „teoretické"

(1.) $B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$, $\det B = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \Rightarrow$

(i) $\Rightarrow$ B je regulární matice, tedy má inverzní

$$B^{-1}: \left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & +1 & -1 & -1 & +2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right) \sim$$

$$\sim \cancel{\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 1 & -4 & \end{array}\right)} \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

(ii) zk.
$$\underbrace{\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}}_{B \cdot B^{-1}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= I} \quad \text{cbd.}$$

(iii) řešení rovnice

$$B \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}: \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B^{-1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$

2) $f(x)=\frac{1-\cos x}{|x|}$ per $x\neq 0$, $f(0)=0$

$$\lim_{x\to 0\pm} f(x) = \lim_{x\to 0\pm} \frac{1-\cos x}{x}\cdot \operatorname{sgn} x \underset{l'H}{=} \lim_{x\to 0\pm} \frac{\sin x}{1}\cdot \operatorname{sgn} x = 0 = f(0)$$

$\Rightarrow$ f è spojitá v bodě 0

$$\left(\text{nebo: } \lim_{x\to 0} \frac{1-\cos x}{|x|} = \lim_{x\to 0} \frac{1-\cos^2 x}{(1+\cos x)\cdot x\cdot \operatorname{sgn} x} = \lim_{x\to 0} \underbrace{\frac{\sin^2 x}{x^2}}_{\to 1}\cdot \underbrace{\frac{x \operatorname{sgn} x}{(1+\cos x)}}_{\to 0} = 0\right)$$

$$f'_\pm(0) \overset{def}{=} \lim_{x\to 0} \frac{1-\cos x}{x\cdot |x|} = \lim_{x\to 0\pm} \frac{1-\cos x}{x^2}\operatorname{sgn} x \underset{l'H}{=} \lim_{x\to 0\pm} \frac{\sin x}{2x}\operatorname{sgn} x =$$

$= \frac{+1}{-2}$ ~~sgn~~ $\Rightarrow$ f nemá v 0 derivaci, ale má $f'_+(0)=\frac{1}{2}$, $f'_-(0)=-\frac{1}{2}$.

3) $f(x)=\cos(e^{2x}-1)$, $a=0$ $\qquad T_2(x)=f(0)+f'(0)\cdot x+\frac{f''(0)}{2}x^2$:

$f(0)=\cos 0=1$

$f'(x)=-\sin(e^{2x}-1)\cdot e^{2x}\cdot 2 \Rightarrow f'(0)=0$

$f''(x)=2\left(e^{2x}\cdot 2\underbrace{(-\sin(e^{2x}-1))}_{=0}-\cos(e^{2x}-1)\cdot e^{2x}\cdot 2\cdot e^{2x}\right)$

$f''(0)=-2\cdot 2=-4$

$\mathcal{J}: T_2(x)=1-\frac{4}{2}\cdot x^2=1-2x^2$